A Division Algebra for Sequences Defined on *all* the Integers

By D. H. Moore

The convolution ring, $\mathcal{S}$, of sequences defined on the nonnegative integers, and the embedding of this ring in a field, have been discussed by Brand [1], Moore [2], [3], Traub [6], and others. Brand [1] specifically mentions that the field in which he embeds $\mathcal{S}$ is a field of ordered pairs of members of $\mathcal{S}$. Traub does not identify his field and does not mention "ordered pairs", but he mentions an analogy to Mikusiński's work [7], and so he probably had in mind the same field of ordered pairs as did Brand. In [2] this writer showed that it was not necessary to create such a field of ordered pairs since there already existed a more natural, less abstract field in which to embed $\mathcal{S}$. It is the purpose of this article to introduce this already existing and more natural field, $\mathfrak{F}$, in which $\mathcal{S}$ may be embedded.

It will be assumed that the reader is familiar with the convolution algebra of sequences as given in [1], [3], and [6] to the point of recognizing $\mathcal{S}$ as an integral domain in which convolution products defined by

$$\{a_\nu\}\ \{b_\nu\} = \left\{\sum_{\mu=0}^{\nu} a_\mu b_{\nu-\mu}\right\} \tag{1}$$

contain no divisors of zero, in which the multiplicative unity is the sequence

$$\{1, 0, 0, 0, \cdots, 0, \cdots\},$$

in which sequences of the form

$$\{c, 0, 0, 0, \cdots, 0, \cdots\}$$

behave like numbers and are identified with numbers:

$$c = \{c, 0, 0, 0, \cdots, 0, \cdots\},$$

in which the sequence

$$\{0, 1, 0, 0, 0, \cdots, 0, \cdots\}$$

is a shift operator denoted by "τ", in which the sequence

$$\{1, 1, 1, \cdots, 1, \cdots\}$$

is a summing operator denoted by "σ", and in which members of $\mathcal{S}$ have operational forms in terms of τ and/or σ.

The sequences σ and τ are related by the equation

$$\sigma(1 - \tau) = 1$$

and since $\mathcal{S}$ has no divisors of zero we introduce fractions and write (for example)

$$\sigma = \frac{1}{1-\tau},$$

$$\frac{1}{\sigma} = 1 - \tau = \{1, -1, 0, 0, 0, \cdots\}.$$

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The fraction $1/\tau$, for example, does not exist as a member of $\mathcal{S}$. But $1/\tau$ will exist as a member of $\mathfrak{F}$.

Let $\mathfrak{F}$ be the class of number valued sequences defined over the integral domain, J, each of which assigns at most a finite number of *nonzero* values to negative integers. For each member of $\mathfrak{F}$ there is a least integer, m, to which the sequence assigns a nonzero value; the sequence will be said to *enter at* m, and the members of $\mathfrak{F}$ will be called *entering* sequences. Equality, sums, and products with numbers, of members of $\mathfrak{F}$ are defined in the usual termwise way. A suggested notation for such a sequence is

$$\{1, 2, 3 \mid 4, 5, 6, \cdots\}$$

where the vertical line—playing a role like a decimal point—separates values assigned to the negative integers from values assigned to the nonnegative integers, and the zeros assigned on the left are omitted for convenience.

Let ν be a variable on J. We define the unit step formula "$u(\nu)$":

$$u(\nu) = \begin{cases} 0, & \nu < 0, \\ 1, & \nu \geqq 0. \end{cases}$$

Then $\{u(\nu)\}$ and $\{(\nu + 1)u(\nu)\}$ (for example) are members of $\mathfrak{F}$, whereas $\{\nu + 1\}$ is not. The braces serve to bind out "ν" converting a formula into a notation for a sequence.

There is a natural one-to-one correspondence between $\mathcal{S}$ (sequences defined on the nonnegative integers) and the subclass, $\mathfrak{F}_0$, of $\mathfrak{F}$ consisting of sequences which enter at nonnegative points:

$$\{a_0, a_1, a_2, \cdots\} \leftrightarrow \{\cdots, 0, 0 \mid a_0, a_1, a_2, \cdots\}. \tag{2}$$

The convolution, or convolution product, of two sequences a and b from $\mathfrak{F}$ is defined by

$$ab = \left\{ \sum_{\mu=-\infty}^{+\infty} a_\mu b_{\nu-\mu} \right\}. \tag{3}$$

If a enters at α or to the right of α and b enters at β or to the right of β, then

$$(ab)_\nu = \begin{cases} \sum_{\mu=\alpha}^{\nu-\beta} a_\mu b_{\nu-\mu}, & \nu \geqq \alpha + \beta, \\ 0, & \nu < \alpha + \beta, \end{cases} \tag{4}$$

$$ab = \left\{ u(\nu - \alpha - \beta) \sum_{\mu=\alpha}^{\nu-\beta} a_\mu b_{\nu-\mu} \right\}. \tag{5}$$

The summation limits are finite in (5) since the sequences are *entering* sequences. In particular, if a and b are members of the subclass $\mathfrak{F}_0$, we may take $\alpha = \beta = 0$ in (5) and (5) becomes

$$ab = \left\{ u(\nu) \sum_{\mu=0}^{\nu} a_\mu b_{\nu-\mu} \right\}. \tag{6}$$

A comparison of (1) and (6) shows that the correspondence (2) is an isomorphism under convolution; we embed $\mathcal{S}$ in $\mathfrak{F}$, identify $\mathcal{S}$ with $\mathfrak{F}_0$, elevate (2) to an equality, and permit any notation for a member of $\mathcal{S}$ to be used as a notation for the corresponding member of $\mathfrak{F}_0$. In particular

$$
\begin{aligned}
1 &= \{\cdots, 0, 0, 0 \mid 1, 0, 0, 0, \cdots, 0, \cdots\} \\
\sigma &= \{\cdots, 0, 0, 0 \mid 1, 1, 1, \cdots, 1, \cdots\} \\
\tau &= \{\cdots, 0, 0, 0 \mid 0, 1, 0, 0, 0, \cdots, 0, \cdots\} \\
\tau^m &= \{\cdots, 0, 0, 0 \mid \underbrace{0, 0, \cdots, 0}_{m \text{ zeros}}, 1, 0, 0, 0, \cdots\}
\end{aligned}
\qquad m = \text{positive integer.} \tag{7}
$$

Defining ζ by:

$$\zeta = \{1 \mid 0, 0, 0, \cdots, 0, \cdots\} \tag{8}$$

we have

$$\zeta^m = \underbrace{\{1, 0, 0, 0, \cdots, 0}_{m \text{ digits}} \mid 0, 0, 0, \cdots, 0, \cdots\} \quad m = \text{positive integer.} \tag{9}$$

Equations (7) and (9) may be verified by induction. Using (5) we may verify that

$$
\begin{aligned}
\tau\zeta &= 1 \\
\tau^{m+n} &= \zeta^m \tau^n \\
\zeta^{m+n} &= \zeta^m \zeta^n \\
\tau^m\{a_\nu\} &= \{a_{\nu-m}\} \\
\zeta^m\{a_\nu\} &= \{a_{\nu+m}\}
\end{aligned}
\qquad m, n \text{ positive integers.}
$$

Under ordinary addition and convolution multiplication $\mathfrak{F}$ is a field. We need only verify here that each nonzero member of $\mathfrak{F}$ has a multiplicative inverse. To begin with, every sequence of the form

$$\{a_0, a_1, a_2, \cdots\}$$

in which $a_0 \neq 0$ (the sequence *enters at the origin*) has an inverse:

$$\{x_0, x_1, x_2, \cdots\}$$

which may be evaluated as follows:

$$
\begin{aligned}
\{a_0, a_1, a_2, \cdots\}\{x_0, x_1, x_2, \cdots\} &= \{1, 0, 0, 0, \cdots\} \\
a_0x_0 &= 1 \\
a_0x_1 + a_1x_0 &= 0 \\
a_0x_2 + a_1x_1 + a_2x_0 &= 0 \\
&\vdots
\end{aligned}
$$

Since the only division involved in solving for the x's is division by a_0, and $a_0 \neq 0$, the x's exist and so the desired inverse exists.

Finally, let a be any nonzero member of $\mathfrak{F}$ which does not enter at the origin.

Since a is an *entering* sequence, there exists a sequence A and a positive integer m such that either

$$a = \tau^m A \quad \text{or} \quad a = \zeta^m A \tag{10}$$

where A enters at the origin, and so has an inverse A^{-1} by the preceding paragraph. Then either

$$(A^{-1}\zeta^m)a = 1 \quad \text{or} \quad (A^{-1}\tau^m)a = 1$$

and so, in any case, a has a multiplicative inverse, and $\mathfrak{F}$ is a field.

Since $\mathfrak{F}$ contains no divisors of zero, products lead to the introduction of fractions:

$$\left.\begin{array}{c} a, b, c \in \mathfrak{F} \\ \text{and} \\ ab = c \\ \text{and} \\ a \neq 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \dfrac{c}{a} \text{ exists as a member of } \mathfrak{F} \\ \text{and} \\ \dfrac{c}{a} = b \\ \text{and} \\ a\left(\dfrac{c}{a}\right) = c. \end{array}\right.$$

In particular

$$\zeta = \{1 \mid 0, 0, 0, \cdots\} = \frac{1}{\tau} = \frac{\{1, 0, 0, 0, \cdots\}}{\{0, 1, 0, 0, 0, \cdots\}}$$

and $1/\tau$ exists as a member of $\mathfrak{F}$.

Members of $\mathfrak{F}$ may be put into operational form in terms of σ, τ, and/or ζ.

Example 1.

$$\begin{aligned} \left\{\frac{\nu(\nu-1)}{2}\,u(\nu+2)\right\} &= \{3, 1 \mid 0, 0, 1, 3, 6, 10, 15, \cdots\} \\ &= \zeta^2\left\{\frac{(\nu-2)\ (\nu-3)}{2}\,u(\nu)\right\} = \frac{\zeta^2}{2}\,\{(\nu^2 - 5\nu + 6)u(\nu)\} \\ &= \frac{\zeta^2}{2}\,(\sigma^2\tau + 2\sigma^3\tau^2 - 5\sigma^2\tau + 6\sigma) \end{aligned}$$

where $\{\nu u(\nu)\} = \sigma^2\tau$ and $\{\nu^2 u(\nu)\} = \sigma^2\tau + 2\sigma^3\tau^2$ as shown in [2], and as may be checked straightforwardly. Then

$$\left\{\frac{\nu(\nu-1)}{2}\,u(\nu+2)\right\} = \sigma^3 - 2\sigma^2\zeta + 3\sigma\zeta^2.$$

In Traub [6, p. 196], every quotient of "generalized" sequences with a nonzero denominator equals a shift operator times an ordinary sequence. Thus, in Traub's notation,

$$\frac{f}{g} = \frac{f}{\omega^i e} = \omega^{-i}\frac{f}{e}$$

where f/e equals an ordinary sequence since e assigns a nonzero value to the origin; ω^{-i} is a shift operator, and is a "generalized" sequence—*an ordered pair of ordinary sequences.* In comparison, in the present paper, we are dealing with entering sequences (defined on J) instead of ordered pairs, and every quotient, b/a, of entering sequences (with nonzero denominator) equals an entering sequence. In evaluating b/a we may replace a, as in (10), by $\tau^m A$ or $\zeta^m A$, as appropriate, and obtain respectively

$$\frac{b}{a} = \zeta^m \frac{b}{A} \quad \text{or} \quad \frac{b}{a} = \tau^m \frac{b}{A}$$

where b/A, ζ^m, and τ^m *are all entering sequences.*

Example 2.

$$\begin{aligned}\frac{\{1, -1, 1, -1, 1, -1, \cdots\}}{\{1, 1, 1 \mid 1, 1, \cdots\}} &= \frac{1/(1+\tau)}{\zeta^3\sigma} = \tau^3 \frac{1-\tau}{1+\tau} \\ &= \tau^3\{1, -2, 2, -2, 2, -2, \cdots\} \\ &= \{0, 0, 0, 1, -2, 2, -2, 2, -2, \cdots\}.\end{aligned}$$

Example 3.

$$\frac{\{3, 1 \mid 0, 0, 1, 3, 6, 10, 15, \cdots\}}{\{0, 0, 0, 1, 3, 3, 1, 0, 0, 0, \cdots\}}$$

$$= \frac{\sigma^3 - 2\sigma^2\zeta + 3\sigma\zeta^2}{\tau^3(\tau+1)^3} \qquad \text{(see example 1)}$$

$$= \frac{\dfrac{1}{(1-\tau)^3} - 2\dfrac{1}{(1-\tau)^2}\dfrac{1}{\tau} + 3\dfrac{1}{1-\tau}\dfrac{1}{\tau^2}}{\tau^3(\tau+1)^3}$$

$$= \left(\frac{6}{\tau^3} - \frac{8}{\tau^4} + \frac{3}{\tau^5}\right)\frac{1}{(1-\tau^2)^3} \qquad \text{(omitting several algebraic steps)}$$

$$= (6\zeta^3 - 8\zeta^4 + 3\zeta^5)\ \{1, 0, 3, 0, 6, 0, 10, 0, 15, 0, \cdots\}$$

(which may be checked by cross multiplication)

$$\begin{aligned}&= \{6, 0, 18 \mid 0, 36, 0, 60, \cdots\} \\ &\quad + \{-8, 0, -24, 0 \mid -48, 0, -80, 0, \cdots\} \\ &\quad + \{3, 0, 9, 0, 18 \mid 0, 30, 0, 45, \cdots\} \\ &= \{3, -8, 15, -24, 36 \mid -48, 66, -80, 105, \cdots\}.\end{aligned}$$

The last result may be checked by cross multiplication:

$$\begin{aligned}&\{0, 0, 0, 1, 3, 3, 1, 0, 0, 0, \cdots\}\{3, -8, 15, -24, 36 \mid -48, 66, -80, 105, \cdots\} \\ &\qquad = \{3, 1 \mid 0, 0, 1, 3, 6, 10, 15, \cdots\}.\end{aligned}$$

A convenient way to multiply two entering sequences is to ignore the vertical lines at first, and then insert a vertical line in the final answer, following rules similar to those for the insertion of a decimal point in a product of decimals.

George Boole's operator, E, [4, p. 16] which shifts a sequence to the left and replaces by zero the terms which pass the origin, operates *only* on sequences which vanish to the left of the origin:

$$E^n\{f(\nu)u(\nu)\} = \{f(\nu + n)u(\nu)\}, \qquad n = \text{nonnegative integer.}$$

Thus E cannot be identified with ζ; neither is E to be discarded, since there is no convolution product to do the job that E does, and that job is important. However, George Boole's symbolic method [4, p. 215] is salvaged if E is replaced by ζ as discussed in [2]. Thus, Boole's symbolic equation [4, pp. 217, 218]

$$\frac{b^x}{E - a} = \frac{b^x}{b - a} + ca^x, \qquad c = \text{arbitrary constant } a, b \text{ numbers}$$

becomes:

$$\frac{\{b^\nu u(\nu)\}}{\zeta - a} = \frac{\{b^\nu u(\nu)\}}{b - a} + \frac{\{a^\nu u(\nu)\}}{a - b}. \tag{11}$$

This follows from the equation

$$\{c^\nu u(\nu)\} = \frac{1}{1 - c\tau} = \frac{\zeta}{\zeta - c}, \qquad c = \text{number}$$

which is easily checked by cross multiplication. To prove (11) we have

$$\frac{\{b^\nu u(\nu)\}}{\zeta - a} = \frac{\zeta}{\zeta - b}\frac{1}{\zeta - a} = \frac{1}{b - a}\frac{\zeta}{\zeta - b} + \frac{1}{a - b}\frac{\zeta}{\zeta - a}$$
$$= \frac{\{b^\nu u(\nu)\}}{b - a} + \frac{\{a^\nu u(\nu)\}}{a - b}.$$

When operational forms of sequences are expressed in terms of ζ they match the Z-transforms of sequences as used, for example, by Aseltine [5] (hence the use of "ζ" for the reciprocal of τ). For example [5, p. 259]

$$\{u(\nu)\} = \sigma = \frac{1}{1 - \tau} = \frac{\zeta}{\zeta - 1}.$$

But now ζ is a sequence and not a variable, a formula in ζ *equals* a sequence rather than being a "transform" of it, and the introduction of the ζ-forms requires no theory of convergence of power series. In [2, pp. 140–143] it is shown that results previously obtained using the theory of functions of a complex variable including branch cuts and the theory of residues, may be obtained by purely algebraic methods from the field properties of $\mathfrak{F}$.

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